Definition (Spin)

$$\mathrm{Spin}(V) := \{ v_1 \cdots v_k \mid v_j \in V \quad |v_j| = 1 \quad \text{even} \} \subset C(V)$$

$$\mathrm{Spin}(m) = \mathrm{Spin}(\mathbb{R}^m)$$

a Lie gp with induced smooth str. from $C(V)$

It is clear $\mathrm{Spin}(V)$ is a group

$v^2 = -1$, $|v| = 1$ in $C(V)$

Proposition: $\forall\ a \in \mathrm{Spin}(m)$,

$$\mathbb{R}^m \ni v \longmapsto a v a^{-1} \in \mathbb{R}^m$$

defines an element $\rho(a) \in SO(m)$.

We have $\rho : \mathrm{Spin}(m) \longrightarrow SO(m)$ morphism of Lie groups

Proof: ① Take $a \in V$, $|a| = 1$, $a^{-1} = -a$

$$-ava^{-1} = ava = (-2\langle a, v\rangle - va)a$$
$$= -va^2 - 2\langle a, v\rangle a$$
$$= v - 2\langle v, a\rangle a$$

$$\rho(a)v := -ava^{-1}$$

$\rho(a)$ acts on $\mathbb{R}^n$ as reflection w.r.t. a

hence $\rho(a) \in O(m)$.

② $\forall\ a = a_1 \cdots a_k \in \mathrm{Spin}(m)$ k even $|a_j| = 1$

$$\rho(a) = \underbrace{\rho(a_1) \cdots \rho(a_k)}_{\text{even number}} \in SO(m)$$

#

Rmk: G_1, G_2 two Lie gps $\Rightarrow$ f is smooth!

Cartan's theorem: $f : G_1 \to G_2$ morphism of gps, continuous

Theorem: For $m \geq 2$,

$$\rho : Spin(m) \longrightarrow SO(m)$$

is a double cover.

Proof: We need to prove 3 things (ρ is already a Lie gp morphism)

① ρ is surjective

② $\ker \rho = \{\pm 1\} \in \mathbb{R} \subset Spin(m)$

③ $Spin(m)$ is connected.

① Note that each isometric in $SO(m)$ is a composition of even-number of reflections on $\mathbb{R}^m$,
$\forall\, a \in \mathbb{R}^m$ $|a| = 1$ $\rho(a) \in O(m)$ is the reflection !

$$\Rightarrow \rho(Spin(m)) = SO(m)$$

② Actually, we prove the following claim :

For $x \in C(\mathbb{R}^m)$, then

$$xv = vx \quad \forall\, v \in \mathbb{R}^m \iff \begin{cases} x \in \mathbb{R} & \text{if } m = \text{even} \\ x \in \mathbb{R} \oplus \mathbb{R} e_1 \cdots e_m & \text{if } m = \text{odd} \end{cases}$$

Proof : " $\Leftarrow$ " $x \in \mathbb{R}$ is clear

When $m = \text{odd}$

$$(e_1 \cdots e_m) e_j = -(-1)^{m-j} e_1 \cdots \hat{e}_j \cdots e_m$$
$$= (-1)^{m-j+j-1} e_j (e_1 \cdots e_m) = e_j e_1 \cdots e_m$$

" $\Rightarrow$ " $x = x_0 + x_1$, $x_0 \in C^+(\mathbb{R}^m)$, $x_1 \in C^-(\mathbb{R}^m)$

Then $\forall\, v \in \mathbb{R}^m$

$$x_0 v = v x_0, \quad x_1 v = v x_1$$

Now we write $x_0 = a_0 + e_1 b_1$

$a_0 \in C^+(\mathbb{R}^m)$ does not contain e_1

$b_1 \in C^-(\mathbb{R}^m)$ does not contain e_1

We take e_1

$$\underbrace{a_0 e_1}_{= e_1 a_0} + e_1 b_1 e_1 = e_1 a_0 + e_1^2 b_1$$

$$\Rightarrow \quad \underbrace{e_1 b_1 e_1}_{= -e_1 b_1} = -b_1 \quad \Rightarrow b_1 = -b_1$$

$$b_1 = 0$$

Iteratively, x_0 does not contain $e_1, e_2, \cdots, e_n$

$$\Rightarrow x_0 \in \mathbb{R}.$$

We write $x_1 = a_1 + e_1 b_0$ $\quad$ $a_1 \in C^-(V)$, $b_0 \in C^+(V)$ do not contain e_1

$$\left.\begin{aligned} e_1 x_1 &= e_1 a_1 - b_0 \\ x_1 e_1 &= \underbrace{a_1 e_1}_{= -a_1} + \underbrace{e_1 b_0 e_1}_{= -b_0} \end{aligned}\right\} \Rightarrow a_1 = 0$$

Then $x_1 = e_1 b_0$ $\quad$ for $\forall\ v \in e_1^\perp$

$$v b_0 = -b_0 v$$

Similarly $\Rightarrow$ $\quad b_0 = e_2 b_0'$

Then $\quad x_1 = e_1 e_2 \cdots e_m \in C^-(V)$ $\quad$ #

③ To show Spin(m) being connected, it is enough to find a path in Spin(m) to connect $\pm 1 \in \ker \rho$.

Since $n \geq 2$, $\quad e_1, e_2 \in \mathbb{R}^m$ $\quad e_1 \perp e_2$ $\quad |e_1| = |e_2| = 1$

$$(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1$$

$$C(V) \ni \exp(t e_1 e_2) = \sum \frac{t^k (e_1 e_2)^k}{k!}$$

$$= \cos t + \sin t\, e_1 e_2$$

$$= \left(\cos\left(\tfrac{t}{2}\right) e_1 + \sin\left(\tfrac{t}{2}\right) e_2\right)\left(-\cos\left(\tfrac{t}{2}\right) e_1 + \sin\left(\tfrac{t}{2}\right) e_2\right)$$

$$\in \mathrm{Spin}(m)$$

$$\gamma : [0, \pi] \longrightarrow \mathrm{Spin}(m)$$
$$t \longmapsto \exp(t e_1 e_2) = \cos t + \sin t\, e_1 e_2$$

$$\gamma(0) = 1 \qquad \gamma(\pi) = -1 . \qquad \#$$

Since $\pi_1(SO(m)) = \mathbb{Z}_2$ for $m \geq 3$

<u>Corollary</u>: For $m \geq 3$, $\mathrm{Spin}(m)$ is a simply connected compact Lie gp.

Example: $\mathrm{Spin}(3) = SU(2) \simeq S^3$ <u>HW3, 5</u>

$\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$

<u>Lemma</u>: Lie algebra $\mathrm{spin}(m)$ of $\mathrm{Spin}(m)$ is given as

$$\Lambda^2 V^* \simeq \mathrm{spin}(m) = \{ e_i e_j : i \neq j \} \subset C^2(V)$$

Let $\rho : \mathrm{spin}(m) \longrightarrow so(m)$ be induced by the morphism $\rho : \mathrm{Spin}(m) \longrightarrow SO(m)$, which is an isomorphism of Lie algebras, and given by for $v \in \mathbb{R}^n$ $a \in \mathrm{spin}(m)$

$$\rho(a) v := [a, v] = av - va \in \mathbb{R}^m .$$

Pf: $\gamma(t) = e^{t e_i e_j} \in \mathrm{Spin}(m) \Rightarrow e_i e_j \in \mathrm{spin}(m)$
$i \neq j$

$$\rho(a) v = \frac{\partial}{\partial t}\Big|_{t=0} e^{ta} v e^{-ta} = [a, v] = av - va . \qquad \#$$

<u>Rmk</u>: used to define Pfaffian

$$so(n) \xrightarrow{\eta} \Lambda^2 (\mathbb{R}^n)^* \xrightarrow{\sigma^{-1}} C^2(\mathbb{R}^m)$$
$$A \longmapsto \eta(A) = \sum_{j,l} \frac{1}{2} \langle e_j, A e_l \rangle e^j \wedge e^l \longmapsto \sum_{j<l} \langle e_j, A e_l \rangle c(e_j) c(e_l)$$

$$\rho^{-1}(A) = \frac{1}{2}\sum_{j<l} \langle Ae_j, e_l\rangle c(e_j)c(e_l)$$

$$\rho^{-1} = -\frac{1}{2}\,\sigma^{-1}\circ\eta$$

IV.3 Spinor (even-dimensional case)

Now suppose V is a real Euclidean space of dimension $m = 2k$

Def: A complex structure J is an element in $O(V)$ s.t. $J^2 = -Id_V$.

$\langle u, v\rangle = \langle Ju, Jv\rangle$

Now consider

$$J = J\otimes Id_{\mathbb{C}} \curvearrowright V\otimes_{\mathbb{R}}\mathbb{C} = V_{\mathbb{C}}$$

$$\Rightarrow \quad V_{\mathbb{C}} = W \oplus \overline{W}$$

where $\begin{cases} W = \{v\in V_{\mathbb{C}} : Jv = \sqrt{-1}\,v\} \\ \overline{W} = \{v\in V_{\mathbb{C}} : Jv = -\sqrt{-1}\,v\}\end{cases}$

Now, for each $v\in V$, we write

$$v = w + \overline{\omega} \quad \text{in } V\otimes_{\mathbb{R}}\mathbb{C} \quad w\in W,\ \overline{\omega}\in\overline{W}$$

Define $c(v)\in \mathrm{End}(\Lambda^{\cdot}\overline{W}^*)$

Given as $c(v) = \sqrt{2}(w^*\wedge - \iota_{\overline{\omega}})$

where $w^*\in\overline{W}^*$ is the metric dual given by

$\langle w^*, \overline{f}\rangle = \langle w, \overline{f}\rangle$ ← $\mathbb{C}$-linear extension of Euclidean metric on V.

for $f\in W$

Note that $\langle\ ,\ \rangle$ induces the Hermitian metric on W & $\overline{W}$ ⑥

s.t. $x = w + \overline{w}$

$$\|x\|^2 = 2\langle w, \overline{w}\rangle = 2h^W(w, w)$$

Proposition: The superspace $S = \Lambda^\bullet \overline{W}^*$ is a Clifford module and

$$C(V) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{End}(S)$$

Moreover, if E is a Clifford module$/\mathbb{C}$, then we have $E = S \otimes F$ (usual tensor)

where $F = \mathrm{Hom}_{C(V)}(S, E)$

$$= \{ f \in \mathrm{Hom}(S, E) : c(v) f = f c(v) \ \forall v \in V \}$$

Proof: ① At first, we show that the action $c(v)$ for $v \in V$ defines a morphism of algebras

$$c : C(V) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{End}(S)$$

It is enough to verify the relations

$$c(x)^2 = -|x|^2$$

Indeed $\forall$ $s \in \Lambda^\bullet \overline{W}^*$

$$c(x)^2 \cdot s = (c(w) + c(\overline{w}))^2 s$$

$$= -2(w^* \wedge \iota_{\overline{w}} + \iota_{\overline{w}} w^*) s$$

$$= -2\langle w, \overline{w}\rangle s = -|x|^2 w$$

② Now we show $C(V)\otimes_{\mathbb{R}}\mathbb{C} \simeq \mathrm{End}(S)$ ⑦

In fact $\dim_{\mathbb{R}} V = m = 2k$

$\Rightarrow \quad \dim_{\mathbb{C}} W = k$

$\dim_{\mathbb{C}} S = 2^k \Rightarrow \dim \mathrm{End}(S) = 2^{2k}$
$= 2^m$
$= \dim C(V)$

So it is enough to show the morphism

$$c : C(V)\otimes\mathbb{C} \longrightarrow \mathrm{End}(S)$$

is injective.

Now $\{w_j\}_{j=1}^k$ ONB of (W, h^W)

Set $\quad e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j) \qquad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j)$

$\{e_j\}_{j=1}^{m=2k}$ ONB of $(V, \langle\, , \rangle) / \mathbb{R}$.

For any $a \in C(V)$

$$c(a) = \sum_{\substack{I \\ =(i_1<\cdots<i_k)}} a_I \, c(e_{i_1}) \cdots c(e_{i_k})$$

Then we can rewrite it in $C(V)\otimes_{\mathbb{R}}\mathbb{C}$ as

$$c(a) = \sum_{\substack{I,J \\ I=(i_1<\cdots<i_k) \\ J=(j_1<\cdots<j_\ell)}} a_{IJ} \, c(w_{i_1}) \cdots c(w_{i_k})\, c(\bar{w}_{j_1}) \cdots c(\bar{w}_{j_\ell})$$

Moreover, if $a_{IJ} = 0 \ \forall I, J \Rightarrow a_I = 0 \ \forall I$ 8

$\Rightarrow a = 0$ in $C(V)$

Now, suppose that $c(a) = 0$ in $\mathrm{End}(S)$

Then

$$c(a) = \sum (-1)^{l} 2^{\frac{k+l}{2}} a_{IJ} \omega_{i_1}^* \wedge \cdots \wedge \omega_{i_k}^* \, e_{\bar{\omega}_{j_1}} e_{\bar{\omega}_{j_2}} \cdots e_{\bar{\omega}_{j_l}}$$

$$\omega_{j_l}^* \wedge \cdots \wedge \omega_{j_1}^* \in S$$

$$c(a)(\omega_{j_l}^* \wedge \cdots \wedge \omega_{j_1}^*) = \sum_{I} (-1)^{l} 2^{\frac{k+l}{2}} a_{IJ} \, \omega_{i_1}^* \wedge \cdots \wedge \omega_{i_k}^*$$

$$= 0$$

$$\Rightarrow \forall I, J \quad a_{IJ} = 0 \quad \Rightarrow a = 0.$$

3 $\mathrm{End}(S)$ is a simple algebra

the only nontrivial irreducible representation is S.

HW 4.1

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